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Closed form adaptive effectiveness factor for numerical models

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Abstract

Concrete does not exhibit rigid-plastic behaviour, but it is nevertheless convenient to assume so for practical design and analysis. In order to account for the actual behaviour, a so-called effectiveness factor is introduced, which have been approximated for various standard problems. For general numerical analysis, however, a different approach is needed, where the actual stress and strain fields are taken into account. Using a recently developed formulation, the implications are analysed for the Mohr-Coulomb yield criterion and a closed-form approximation for use in finite element limit analysis is proposed. The reduced yield surface is compared to the closed-form approximation for various choices of reinforcement layouts.

Keywords: yield surface, effectiveness factor, optimisation, finite element

1 Introduction

Assuming perfectly plastic material behaviour is often convenient when it comes to practical design and analysis. This assumption simplifies the material model greatly and for some materials, it provides a decent approximation. Combined with the assumption of a rigid material, i.e. no elastic deformations, we arrive at the well-known and widely used limit analysis, which provides an elegant framework for handling many kinds of problems in structural and geotechnical engineering, e.g. plate bending (Johansen 1962), membrane action (Nielsen & Hoang 2010), and slope stability (Chen 2013).

The behaviour of concrete does not resemble a rigid-plastic material, but reinforced concrete structures are nevertheless treated within the framework of limit analysis (Nielsen & Hoang 2010). The reinforcement typically behaves close to the rigid-plastic assumption, however, special care is needed when the concrete is governing the behaviour. Conventionally this is done by adjusting the design strength of the concrete by a so-called effectiveness factor (Nielsen & Hoang 2010) - an approach which standards, e.g. the Eurocode (European Committee for Standardization 2005), have adopted. The scope of the effectiveness factor is to account for the actual material behaviour.

Ideally the effectiveness factor should be determined based on experiments for every problem. This is of course not feasible, hence, guidelines, i.e. formulas, are typically given in the standards based on experimental programs for various standard problems, e.g. shear in beams (Nielsen & Hoang 2010). This approach, however, is incompatible with numerical methods since the effectiveness factor is applied on the model level and does not account for the actual stress and strain fields. Moreover, the design engineer have to chose what problem, he is dealing with, to get an approximation of the effectiveness factor, while the actual problem at hand may comprise several of these standard problems.

Today, numerical methods are used widely in practice. Methods such as the finite element method are essential in the analysis and design of complex structures, however, the current concept of the effectiveness factor (as e.g. adopted in the current Eurocode 2) does not fit general numerical methods. This is especially true when considering finite element limit analysis (FELA), a special case of the finite element method where a rigid-plastic behaviour is assumed. FELA is widely used in geotechnical engineering (Sloan 1988, Makrodimopoulos & Martin 2005), but also for concrete structures (Larsen 2010, Herfelt et al. 2017). For FELA, an adaptive

approach to the effectiveness factor is needed.

It is experimentally well-known that the compressive strength of concrete is dependent on the level of transverse strains (Demorieux 1969, Robinson & Demorieux 1977). Positive strain, i.e. tension, perpendicular to the direction of the principal compression stresses induces cracking and reduces the effective strength of the material. Several empirical equations have been developed in the last decades to account for this phenomenon (Vecchio & Collins 1986, Belarbi & Hsu 1995, Fehling et al. 2008, Zhang 1997, Kollegger 1988, Hoang et al. 2012, *fib* Model Code 2010 2013, Kaufmann 1998, Muttoni 1989). In this work, the resulting yield surface is analysed for a Mohr-Coulomb material. Based on the observations, a closed-form convex approximation is proposed which will fit the format of finite element limit analysis.

2 Effectiveness factor

Using the formulation of e.g *fib* Model Code 2010 (2013), the effectiveness factor, ν , can be split into two factors:

$$\nu = \eta_{f_c} \eta_{\epsilon} \quad (1)$$

The first factor, η_{f_c} , originally proposed by Muttoni (1989), accounts for the actual stress-strain relationship and is given as

$$\eta_{f_c} = \min \left\{ \left(\frac{30 \text{ MPa}}{f_c^0} \right)^{1/3}, 1 \right\} \quad (2)$$

where f_c^0 is the uniaxial compressive strength. If f_c^0 is larger than 30 MPa, the strength is reduced accordingly. The second factor, η_{ϵ} , is related to the state of strain. Cracks parallel to the compressive stresses reduce the effective strength of the material, which η_{ϵ} accounts for. In many of the aforementioned references, including the *fib* Model Code 2010 (2013), the strain dependency of the effectiveness factor follows a hyperbolic variation, which can be formulated as follows:

$$\eta_{\epsilon} = \min \left\{ \frac{1}{c_1 + c_2 \epsilon_1}, c_3 \right\} \quad (3)$$

where c_1 , c_2 , and c_3 are empirical constants. In this paper, $c_1 = 1$, $c_2 = 80$, and $c_3 = 1$ are chosen for analysis of the numerical concept. It should be noted that other values of c_1 , c_2 , and c_3 have been proposed in the aforementioned references based on experimental fitting. For the demonstration of the numerical concept, however, the specific values are of lesser importance. The adopted relationship between the principal strain and η_{ϵ} is illustrated in Fig. 1.

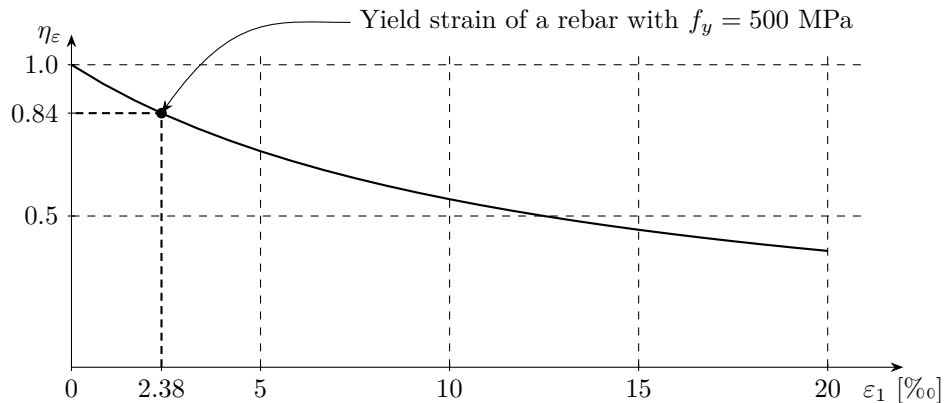


Fig. 1: Relationship between ϵ_1 and η_{ϵ} for $c_1 = 1$, $c_2 = 80$, and $c_3 = 1$.

Fig. 1 shows that the effectiveness factor decrease rapidly. At 2.38 ‰ corresponding to the yield strain of typical mild steel, the effective strength is reduced by approximately 16 %. As the cracks open and the yielding reinforcement undergo plastic deformations, the strength will decrease even further. In this work, the effectiveness factor is used in combination with the Mohr-Coulomb yield criterion commonly used for concrete. The scope is to visualise the resulting yield surface for reinforced concrete panels subjected to in-plane conditions and approximate it with a simpler function suitable for finite element limit analysis.

3 Reduced yield surface

3.1 Optimisation

The reduced yield surface is calculated using non-linear optimisation. For a given stress state, σ , a multiplier is determined which projects the stress state unto the yield surface, which can be stated as an optimisation problem. General non-linear optimisation problems has the form (Nocedal & Wright 2006):

$$\begin{aligned} & \text{minimise} && f(\mathbf{x}) \\ & \text{subject to} && g_i(\mathbf{x}) = 0, \quad i = 1, \dots, m_e \\ & && h_j(\mathbf{x}) \leq 0, \quad j = 1, \dots, m_i \end{aligned} \quad (4)$$

where \mathbf{x} are the problem variables and m_e and m_i are the number of equality constraints and inequality constraints, respectively. The functions f , g_i , and h_j can be general non-linear and concave, which makes non-linear optimisation problems (4) notoriously hard to solve due to the possible existence of multiple local minima and stationary points.

3.2 Governing equations for the yield surface

As mentioned in the introduction we assume a plane stress state. Moreover, we only consider orthogonal reinforcement and the two directions are denoted x and y . The effective stresses, σ , are split into concrete and reinforcement stresses, σ_c and σ_s :

$$\sigma = \sigma_c + \rho_s \sigma_s \quad (5)$$

with

$$\sigma_c = \begin{bmatrix} \sigma_x^c \\ \sigma_y^c \\ \tau_{xy}^c \end{bmatrix}, \quad \sigma_s = \begin{bmatrix} \sigma_x^s \\ \sigma_y^s \\ 0 \end{bmatrix}, \quad \rho_s = \begin{bmatrix} \frac{A_{sx}}{t} & & \\ & \frac{A_{sy}}{t} & \\ & & 0 \end{bmatrix}, \quad (6)$$

where A_{sx} and A_{sy} are the reinforcement areas per unit length in the x and the y -directions, respectively, and t is the thickness. As (5) shows it is assumed that the reinforcement only carries normal stresses. Moreover, it is assumed that the reinforcement only tension with a linear-elastic perfectly plastic stress-strain relationship, i.e. the relationship between the reinforcement stress and the strain is given as

$$\sigma_x^s = \begin{cases} 0 & \text{if } \varepsilon_x < 0 \\ E_s \varepsilon_x & \text{if } 0 \leq \varepsilon_x \leq f_y/E_s \\ f_y & \text{if } f_y/E_s < \varepsilon_x \end{cases} \quad (7)$$

and identically for the y -direction. Eq. 7 does not define a unique relationship however, hence, it is necessary to consider Mohr's circle, which provides two additional constraints. The centre of Mohr's circle is given as

$$C = \frac{\varepsilon_1 + \varepsilon_2}{2} = \frac{\varepsilon_x + \varepsilon_y}{2} \quad (8)$$

which gives the first relation between the principal strains and reinforcement strains. The radius of Mohr's circle gives the second relation:

$$r = \frac{\varepsilon_1 - \varepsilon_2}{2} = \frac{\varepsilon_x - \varepsilon_y}{2 \cos 2\theta} \quad (9)$$

where θ is the angle of the first principal stress (and strain), which can be computed as

$$\theta = \frac{1}{2} \arctan \left(\frac{2\tau_{xy}^c}{\sigma_x^c - \sigma_y^c} \right) \quad (10)$$

Rearranging the terms and inserting expression for 2θ into the cosine gives

$$\cos 2\theta = \pm \frac{1}{2} \frac{\sigma_x^c - \sigma_y^c}{\varphi} \quad (11)$$

with

$$\varphi = \sqrt{\left(\frac{\sigma_x^c - \sigma_y^c}{2} \right)^2 + (\tau_{xy})^2} \quad (12)$$

The concrete stresses should satisfy the Mohr-Coulomb criterion, however, in this case the compressive strength, f_c , is a function of the strains, ε . For plane stress, the criterion can be stated as

$$\begin{aligned} \sigma_1^c &\leq f_t \\ k\sigma_1^c - \sigma_2^c &\leq f_c(\varepsilon) \\ -\sigma_2^c &\leq f_c(\varepsilon) \end{aligned} \quad (13)$$

with

$$k = \left(\sqrt{\mu^2 + 1} + \mu \right)^2, \quad \mu = \tan \phi \quad (14)$$

where ϕ is the internal angle of friction. For concrete, $\phi = 37^\circ$ is commonly used corresponding to $k = 4$. Introducing three auxiliary variables,

$$\sigma_m = \frac{\sigma_x^c + \sigma_y^c}{2}, \quad \sigma_d = \frac{\sigma_x^c - \sigma_y^c}{2}, \quad \varphi = \sqrt{(\sigma_d)^2 + (\tau_{xy})^2}, \quad (15)$$

the criterion (13) can now be stated as

$$\begin{aligned} \sigma_m + \varphi &\leq f_t \\ (k-1)\sigma_m + (k+1)\varphi &\leq f_c(\varepsilon) \\ -\sigma_m + \varphi &\leq f_c(\varepsilon) \end{aligned} \quad (16)$$

The definition of φ in (15) is identical to (12), hence, the same variable can be used. The compressive strength is given as

$$f_c(\varepsilon) = \eta_\varepsilon(\varepsilon) \eta_{f_c} f_c^0 \quad (17)$$

where f_c^0 is the initial strength. The scalar, η_{f_c} , is constant (see Eq. 2) and therefore not a part of the optimisation, while $\eta_\varepsilon(\varepsilon)$ is a function of the first principal strain:

$$\eta_\varepsilon(\varepsilon) = \min \left\{ \frac{1}{c_1 + c_2 \varepsilon_1}, c_3 \right\} \quad (18)$$

In this work, $c_1 = 1$, $c_2 = 80$, and $c_3 = 1$ are used. In order to define the state of strains uniquely, a relationship between the second principal concrete stress and second principal strain

is needed. For this purpose, we assume that the concrete is in the elastic range or just at the on-set of plastic yielding, i.e. no plastic deformations have taken place. A linear elastic relationship can therefore be assumed:

$$\sigma_2^c = -\sigma_m + \varphi = E_c \varepsilon_2 \quad (19)$$

where E_c is the Young's modulus of concrete.

The necessary equations have now been established. The system is implemented as a general non-linear optimisation problem and solved using the built-in sequential quadratic programming solver in Matlab.

3.3 Closed-form approximation

The closed-form approximation should be convex and should fit the format of second-order cone programming which is commonly used for finite element limit analysis. Again, we assume that the concrete is in the elastic range or just at the on-set of plastic deformations, hence, the second principal strain can be approximated as

$$\varepsilon_2 = -\frac{\eta_{f_c} f_c^0}{E_c (c_1 + c_2 \varepsilon_1)} \quad (20)$$

Inserting this into (8) gives

$$\varepsilon_1 - \frac{\eta_{f_c} f_c^0}{E_c (c_1 + c_2 \varepsilon_1)} - \varepsilon_x - \varepsilon_y = 0 \quad (21)$$

Assuming that the reinforcement in one direction is at the on-set of yielding, i.e.

$$\varepsilon_x = f_{yx}/E_s, \quad \varepsilon_y = 0, \quad \text{or} \quad \varepsilon_y = f_y/E_s, \quad \varepsilon_x = 0, \quad (22)$$

(21) can be rearranged to obtain

$$\varepsilon_1 = \frac{c_2 \varepsilon_i - c_1 \pm \sqrt{c_1^2 + c_2^2 \varepsilon_i^2 + 2c_1 c_2 \varepsilon_i + 4c_2 \eta_{f_c} f_c^0 / E_c}}{2c_2} \quad (23)$$

which gives an expression of ε_1 - and η_ε - solely based on the characteristics of the reinforcement and concrete. The effectiveness factor should be active when both large compressive stresses in the concrete and tension in the reinforcement are present, hence, cases such as uniaxial compression should be unaffected. In order to ensure this, the reduction is enforced accordingly:

$$\frac{\sigma_i^s}{f_{yi}} \leq \frac{\eta_{f_c} f_c^0 - \sigma_2}{\eta_{f_c} f_c^0 - \eta_\varepsilon \eta_{f_c} f_c^0} = \frac{1}{1 - \eta_\varepsilon} \left(1 - \frac{\sigma_2}{\eta_{f_c} f_c^0} \right) \quad (24)$$

where σ_i^s is the reinforcement stress in the i th direction. This criterion is implemented in the conventional convex formulation of the Mohr-Coulomb criterion (see e.g. Bisbos & Pardalos 2007, Krabbenhøft et al. 2007).

4 Analysis

The largest reductions in strength will occur when one direction is reinforced heavily compared to the other. Therefore, three combinations of reinforcement ratios are analysed, namely $\Phi_y/\Phi_x = 1, 4$, and 10 , where the reinforcement ratio is defined as

$$\Phi_i = \frac{A_{si} f_{yi}}{t f_c^0} \quad (25)$$

For the presented analysis, the following material parameters are used:

$$f_c^0 = 20 \text{ MPa}, f_t = 0, k = 4, t = 200 \text{ mm}, A_{sx} = 500 \text{ mm}^2/\text{m}, E_c = 30 \text{ GPa}, E_s = 210 \text{ GPa}$$

For the purpose of the analysis, the yield strength is varied between 250 MPa and 1500 MPa, while the reinforcement in the y -direction is varied between 500 mm²/m and 5000 mm²/m. For reference, two layers of Ø6 rebars per 200 mm corresponds to a reinforcement area of 503 mm²/m.

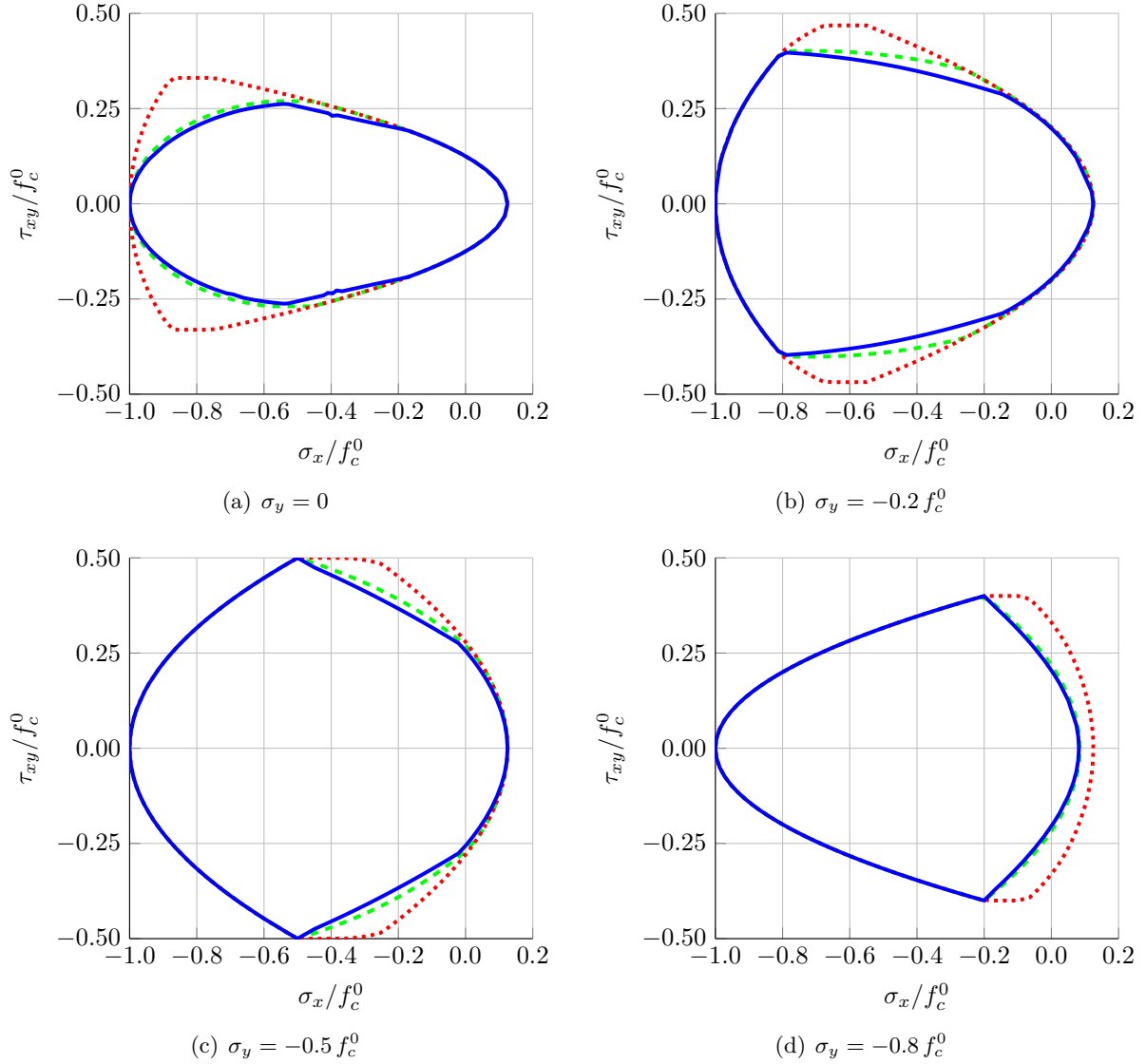


Fig. 2: Dotted red: original Mohr-Coulomb envelope, solid blue: reduced Mohr-Coulomb envelope using non-linear optimisation, dashed green: closed form approximation. Using $f_c^0 = 20$ MPa, $f_y = 1000$ MPa, $A_{sx} = 500$ mm²/m, $A_{sy} = 500$ mm²/m.

The reduced yield surface calculated using non-linear optimisation as well as the original Mohr-Coulomb yield surface (i.e. without the effectiveness factor) are shown in Fig. 2 together with the closed-form approximation. As expected the effect of the effectiveness factor can be seen where large compressive stresses in the concrete and tension in the reinforcement are present. Fig. 2 shows that closed-form approximation captures the reduction rather well, however, it is slightly on the unsafe side in some cases compared to the non-linear optimisation.

The reduced envelope calculated using non-linear optimisation shows, that the formulation leads to a slightly concave shape, which can be approximated reasonably well as a convex envelope.

Moreover, it is observed that when both principal concrete stresses are in compression, the effectiveness factor is one, which is one of the scopes for this adaptive approach.

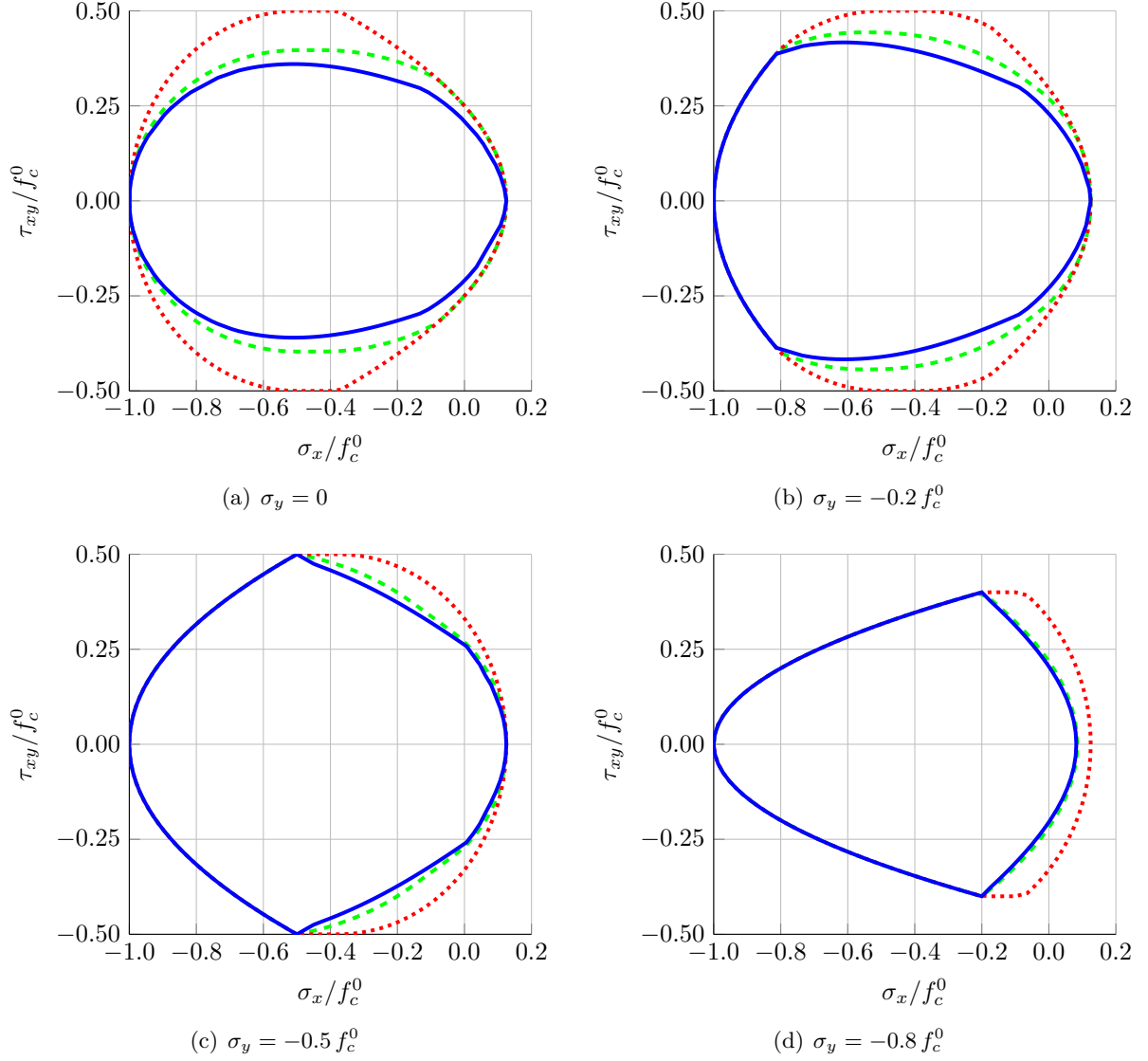


Fig. 3: Dotted red: original Mohr-Coulomb envelope, solid blue: reduced Mohr-Coulomb envelope using non-linear optimisation, dashed green: closed form approximation. Using $f_c^0 = 20$ MPa, $f_y = 1000$ MPa, $A_{sx} = 500$ mm²/m, $A_{sy} = 2000$ mm²/m.

Fig. 3 shows the same tendencies as Fig. 2, however, it is observed that the closed-form approximation is better in Fig. 2. It is expected that the accuracy is highest when the rebars in the two directions are equal. Compared to the original Mohr-Coulomb yield surface, a considerable reduction is seen in Fig. 3(a) where the maximum shear stress is reduced from $0.5f_c$ to approximately $0.36f_c$, corresponding to a reduction of 28 %.

Fig. 4 shows cuts through the yield surface along the diagonal where $\sigma_x = \sigma_y$ for varying reinforcement ratios. It is clearly seen that effectiveness factor decreases as the reinforcement areas increase, increasing the discrepancies between the reduced yield surface and the original yield surface. Again, it is observed that the yield envelope calculated using non-linear optimisation is concave near the apex, i.e. at $\sigma_x = \sigma_y = \tau_{xy} = 0.5f_c^0$.

Fig. 5 shows the most extreme case analysed: The reinforcement area in the y -direction is 10 times larger than the reinforcement area in the x -direction. In Fig. 5(d), the effect of the effectiveness factor is apparent as the tensile capacity in the y -direction is reduced substantially,

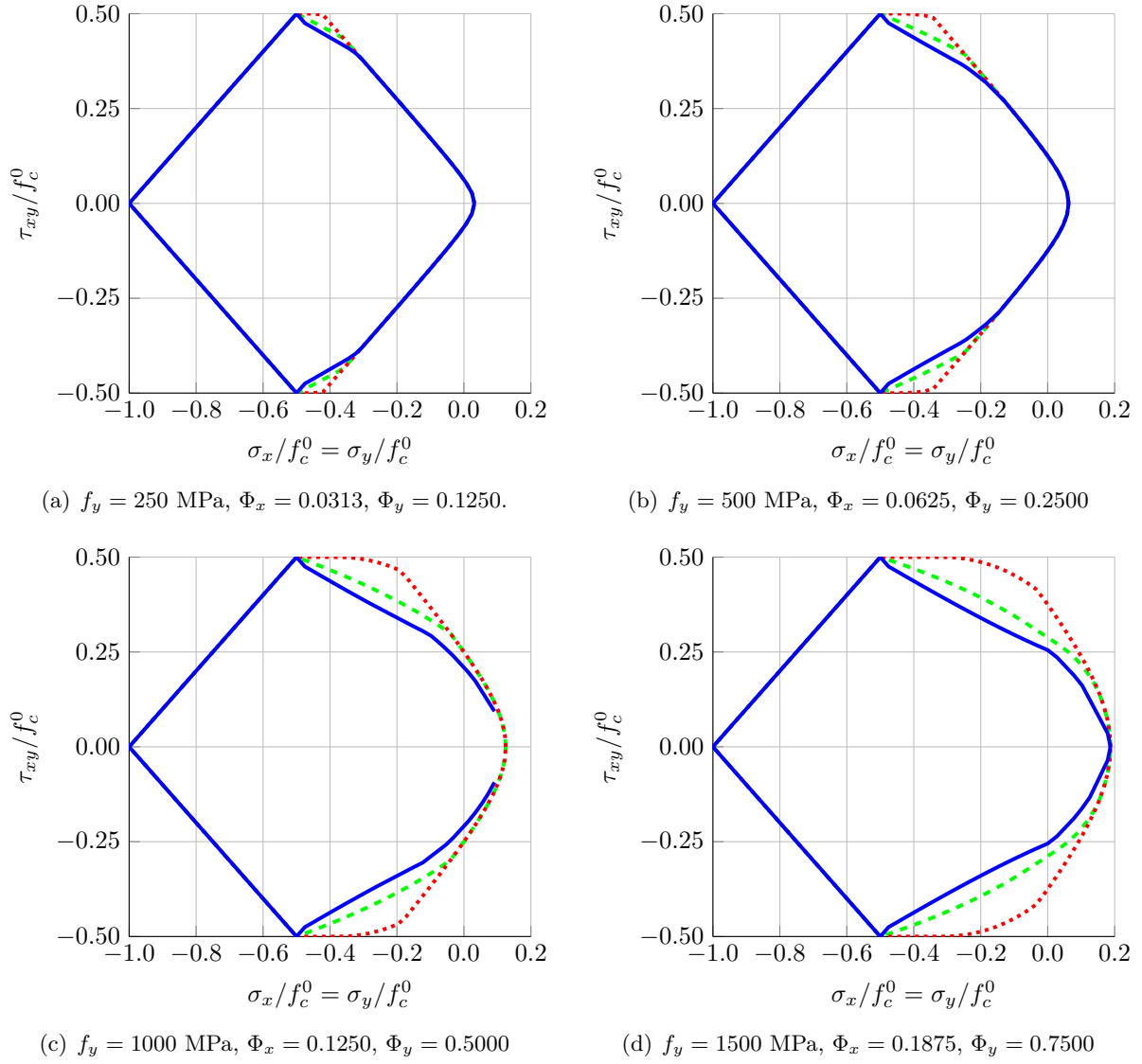


Fig. 4: Dotted red: original Mohr-Coulomb envelope, solid blue: reduced Mohr-Coulomb envelope using non-linear optimisation, dashed green: closed form approximation. $f_c^0 = 20$ MPa, $A_{sx} = 500$ mm²/m, $A_{sy} = 2000$ mm²/m.

and yielding of the rebars cannot be achieved while maintaining a compressive stress in the x -direction of $-0.8f_c^0$.

The closed-form approximation provides, nevertheless, a decent approximation. Especially in the aforementioned case of $\sigma_x = -0.8f_c^0$, the closed-form approximation captures the reduction rather well. The approximation is, however, still slightly on the unsafe side compared to the non-linear optimisation.

In Fig. 5(c) it appears that the yield surface calculated using non-linear optimisation follows the unreinforced envelope indicated by a dotted magenta line. At the apex, the concrete behaves as unreinforced since it is assumed that the reinforcement only carries tension. Activating the reinforcement requires tensile strains, which reduces the effective concrete strength, and it is therefore more beneficial to follow the unreinforced curve near the apex. At some point, however, the reinforcement can be activated and the curve diverges from the unreinforced envelope. This behaviour also suggests that at least two local optima exist near the apex, making the optimisation problem difficult to solve.

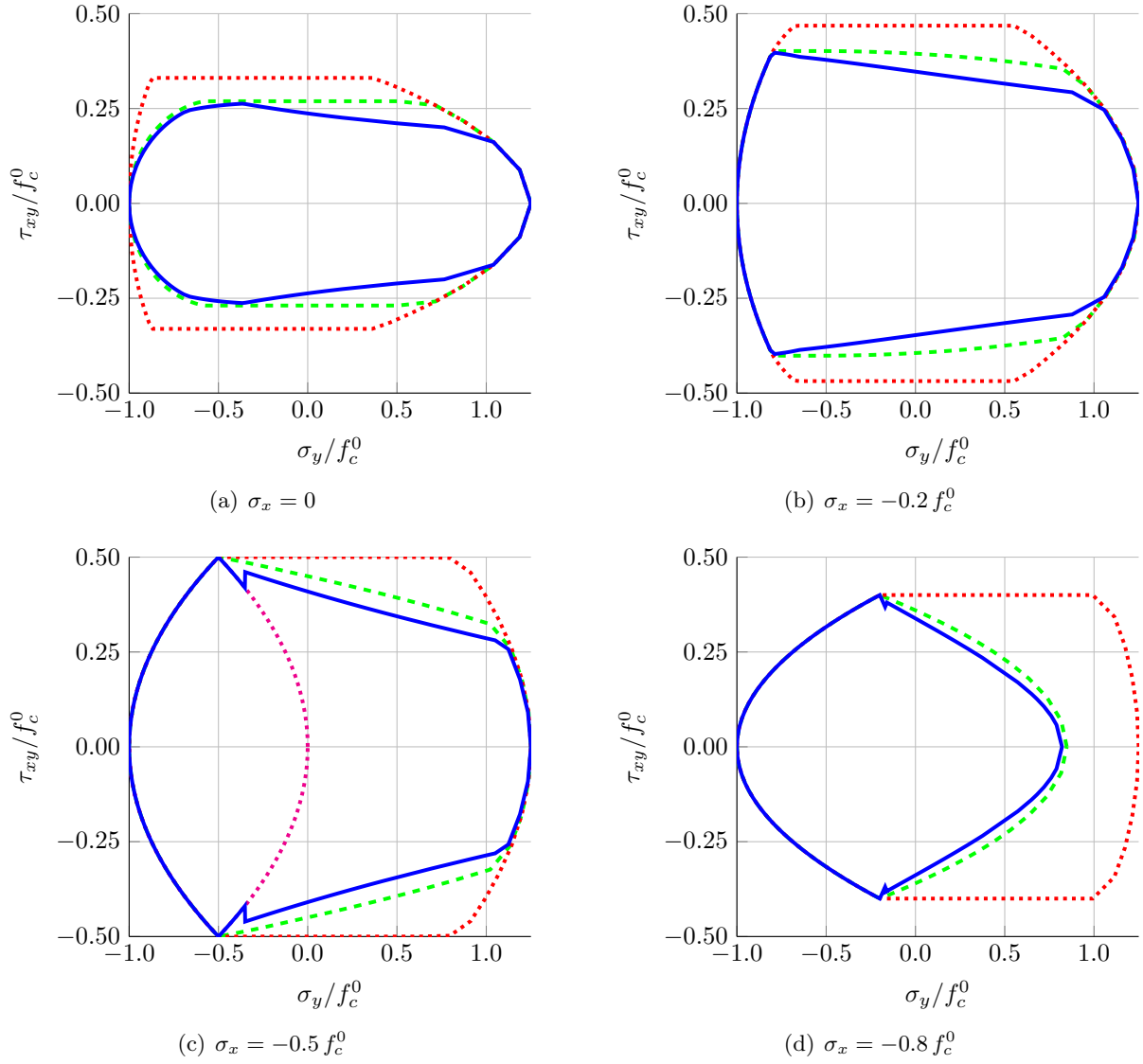


Fig. 5: Dotted red: original Mohr-Coulomb envelope, solid blue: reduced Mohr-Coulomb envelope using non-linear optimisation, dashed green: closed form reduced Mohr-Coulomb envelope approximation. $f_c^0 = 20$ MPa, $f_y = 1000$ MPa, $A_{sx} = 500$ mm²/m, $A_{sy} = 5000$ mm²/m.

5 Conclusions

The use of idealised rigid-plastic material model in concrete mechanics requires careful consideration of the effective strength of the material. It is well known that cracking parallel to the compressive strut decrease the compressive strength, however, this is currently implemented in the various standards on a case-by-case basis.

In the *fib* model code 2010 an effectiveness factor based on the state of strains has been suggested. This facilitates an adaptive approach suitable for numerical calculations. In order to investigate the resulting yield surface, the empirical formula for the effectiveness factor is implemented for the Mohr-Coulomb criterion using non-linear optimisation. Moreover, a closed-form approximation is suggested based on convex optimisation.

From the conducted analysis, the effect of this effectiveness factor is apparent. Large shear stresses combined with compression reduce the strength, while uniaxial compression does not. The non-linear optimisation problem is slightly concave and has several local optima, which makes it a rather challenging problem. The closed-form approximation, on the other hand, is

convex and provides a decent approximation. In cases when one direction is reinforced heavily compared to the other, the closed-form approximation slightly overestimates the strength compared to the non-linear optimisation implementation. This can be corrected by using a global reduction factor in addition to the effectiveness factor, and thereby ensuring a safe solution.

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